

Forcing with quotients*

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Abstract

We study an extensive connection between factor forcings of Borel subsets of Polish spaces modulo a σ -ideal and factor forcings of subsets of countable sets modulo an ideal.

1 Introduction

In recent years there has been a wave of interest in partial orders given as quotients, either $\text{Borel}(X)/I$ for a σ -ideal I on a Polish space X , or $\text{Power}(X)/J$ for an ideal J on a countable set X . The former turned out to be very close to traditional forcings adding a real, and they allow of a comprehensive theory [16]. The situation for the latter is much less clear and less well understood. In this paper we describe a close relationship between the two classes of posets. The connecting link is the following definition due to Brendle:

Definition 1.1. For a σ -ideal I on 2^ω or ω^ω the *trace ideal* $\text{tr}(I)$ on $2^{<\omega}$ or $\omega^{<\omega}$ is defined by $a \in \text{tr}(I) \leftrightarrow \{r \in 2^\omega : \exists^\infty n \ r \restriction n \in a\} \in I$.

We prove

Theorem 1.2. *Suppose that I is a σ -ideal on $X = 2^\omega$ or $X = \omega^\omega$. If $P = \text{Borel}(X)/I$ is a proper forcing with continuous reading of names, then $Q = \text{Power}(2^{<\omega} \text{ or } \omega^{<\omega})/\text{tr}(I)$ is a proper forcing as well and in fact Q is naturally isomorphic to a two step iteration of P and an \aleph_0 -distributive forcing.*

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This result makes it easy to generate and understand a large variety of factors of ideals on ω . Our methods provide many ideals for which these factors are proper as well as examples of ideals for which the factor forcings are improper.

Earlier results in this area include a note of Steprāns [14] on what in retrospect are trace ideals for a small class of forcings, a result of Balcar, Hernández, and Hrušák [1] regarding the properness of the factor $\text{Power}(\mathbb{Q})/\text{nowhere dense sets}$, and results of Steprāns and Farah concerning the properness of factors $\text{Power}(\omega)/J$ for various analytic P-ideals J . It should be noted that the trace ideals are analytic P-ideals only in the case the original forcing P_I had an exhaustive submeasure on it by a result of Solecki [13].

The notation used in the paper follows the set theoretic standard of [6]. If I is a σ -ideal on a Polish space X , the symbol P_I denotes the poset of I -positive Borel subsets of X ordered by inclusion. If J is an ideal on a countable set X , the symbol Q_J denotes the poset of J -positive subsets of X ordered by inclusion. For a tree $T \subset (2 \times \omega)^{<\omega}$ the symbol $[T]$ denotes the set of all its infinite branches and the symbol $p[T]$ its projection, that is the set of those $r \in 2^\omega$ such that there is $b \in \omega^\omega$ such that the pair r, b constitutes a branch through the tree T . The characteristic function of a set $a \subset \omega$ is denoted by $\chi(a)$. For a sequence $t \in 2^{<\omega}$ the symbol $[t]$ denotes the basic open subset of the space 2^ω determined by t . LC denotes the use of suitable large cardinal assumptions.

2 The continuous reading of names

The following natural definition is critical for the wording and proofs of all results in this paper.

Definition 2.1. Let I be a σ -ideal on a Polish space. The forcing P_I has the *continuous reading of names* if for every I -positive Borel set B and a Borel function $f : B \rightarrow 2^\omega$ there is an I -positive Borel set $C \subset B$ such that the function f is continuous on it.

There are several natural restatements of this property.

Claim 2.2. Let I be a σ -ideal on a Polish space X . The following are equivalent:

1. the forcing P_I has the continuous reading of names
2. for every I -positive Borel set B and a countable collection $\{D_n : n \in \omega\}$ of Borel sets there is an I -positive Borel set $C \subset B$ such that all sets $D_n \cap C$ are relatively open in C
3. for every I -positive Borel set B and every Borel function $f : B \rightarrow Y$ to a Polish space Y there is an I -positive Borel set $C \subset B$ such that $f \upharpoonright C$ is continuous.

Proof. (1) \rightarrow (2). Fix sets $B, D_n : n \in \omega$ and define a Borel function $f : B \rightarrow 2^\omega$ by $f(r)(n) = 1$ if $r \in D_n$. By the continuous reading of names there is an

I -positive Borel set $C \subset B$ such that $f \upharpoonright C$ is continuous. It is immediate that the sets $D_n \cap C$ must be relatively open in C .

(2) \rightarrow (3). Suppose that B is a Borel I -positive set and $f : B \rightarrow Y$ is a Borel function. For every basic open set O from some fixed countable basis for the space Y , let $D_O = f^{-1}O$. It is clear that D_O is a Borel set and if $C \subset B$ is any set such that all sets $D_O \cap C$ are relatively open in C , the function $f \upharpoonright C$ must be continuous.

(3) \rightarrow (1). Trivial. \square

Most definable proper partial orderings have the continuous reading of names under a suitable representation.

Example 2.3. [16] 2.2.3. Every proper ω^ω -bounding poset P_I has the continuous reading of names.

Proof. For simplicity assume that the underlying space of the ideal I is 2^ω . Suppose B is a Borel I -positive set and $f : B \rightarrow 2^\omega$ is a Borel function. Let $T \subset (2 \times 2 \times \omega)^{<\omega}$ be a tree which projects to the graph of the function f . By a standard absoluteness argument, $B \Vdash \text{“for some } \dot{s} \in 2^\omega, \dot{b} \in \omega^\omega \text{ the triple } \langle \dot{r}_{gen}, \dot{s}, \dot{b} \rangle \text{ constitutes a branch through the tree } \tilde{T} \text{”}$. Since the forcing P_I is bounding, there is a condition $D \subset B$ which forces \dot{b} to be pointwise dominated by some function $c \in \omega^\omega$. Let S be the subset of the tree T consisting of those sequences whose third coordinate is pointwise dominated by the function c . Then S is a finitely branching tree and

- $p[S]$ is a compact subset of the graph of the function f , so it is a graph of a continuous subfunction of f
- $C = pp[S]$ is a compact subset of the set B , D forces the generic real into \dot{C} and therefore C is I -positive.

All in all, $C \subset B$ is an I -positive compact set on which the function f is continuous. \square

Example 2.4. If the ideal I is σ -generated by closed sets then P_I has the continuous reading of names.

Proof. Let us first deal with the case of the σ -ideal generated by nowhere dense sets. It is a classical result [7] 8.38 that for every Polish space X and every Borel function $f : X \rightarrow \omega^\omega$ there is a comeager G_δ set $C \subset X$ such that f is continuous on it.

In the general case, suppose that B is a Borel I -positive set and $f : B \rightarrow \omega^\omega$ is a Borel function. By a result of Solecki [13], thinning out the set B we may assume that it is G_δ and for every basic open set O , $O \cap B \notin I \leftrightarrow O \cap B \neq \emptyset$. It immediately follows that then every closed set in the ideal I is nowhere dense in the set B . Since the set B is G_δ , it is Polish in the relative topology, and so every set $C \subset B$ comeager in B must be positive in the ideal I . By the first paragraph of the proof, there must be a comeager G_δ set $C \subset B$ such that the function f is continuous on it. \square

These two classes of examples include many forcings used in practice, such as the Cohen, Solovay, or Miller reals. In other situations, the continuous reading of names has to be checked carefully.

Example 2.5. The Laver forcing in the natural presentation has the continuous reading of names.

Example 2.6. The *Steprāns forcing* [15] in the natural presentation does not have the continuous reading of names. Here the Steprāns forcing P_I is obtained from a Borel function $f : 2^\omega \rightarrow 2^\omega$ which cannot be decomposed into countably many continuous functions by considering the ideal I σ -generated by the sets on which the function f is continuous. The poset P_I is proper and up to the forcing equivalence does not depend on the initial choice of the function f —see [16], 2.3.49.

It is interesting to note that in a slightly different presentation the ideal associated with Steprāns forcing is generated by closed sets and therefore does have the continuous reading of names. Let us describe this different presentation.

We will need a definite example of a Borel function which cannot be decomposed into countably many continuous functions, due to Pawlikowski [3]. Consider the space $\omega + 1$ equipped with the order topology, the space $(X, \sigma) = (\omega + 1)^\omega$ with the product topology, and the *Pawlikowski function* $P : X \rightarrow \omega^\omega$ defined by $P(r)(n) = r(n) + 1$ if $r(n) \in \omega$ and $P(r)(n) = 0$ if $r(n) = \omega$. This function cannot be decomposed into countably many continuous functions and it is in a sense a minimal such example [12].

Let I be the σ -ideal on the space X generated by the sets B on which the function P is continuous. Clearly the poset P_I does not have the continuous reading of names as witnessed by the function P . However, the function P turns out to be the only obstacle. Namely, if the space X is equipped with the smallest Polish topology $\tau \supset \sigma$ which makes the function P continuous and generates the same Borel structure, the σ -ideal I is generated by τ -closed sets and so the forcing P_I has the continuous reading of names in this new presentation. An outline of the easy argument: the topology τ is the product topology on X if $\omega + 1$ is viewed as a discrete space. If $B \subset X$ is a set such that $P \upharpoonright B$ is continuous with respect to the topology σ and C is the τ -closure of the set B , then $P \upharpoonright C$ is continuous with respect to σ as well. If U, V are basic open subsets of (X, σ) and ω^ω respectively such that $P''(B \cap U) \subset V$ then $P''(C \cap U) \subset V$ as well.

Definition 2.7. Let J be an ideal on ω . The *Prikry forcing* $P(J)$ for the ideal J is defined as the set of all pairs $\langle t, a \rangle$ where $t \subset \omega$ is a finite set, $a \subset \omega$ is a set in the ideal J , and $\langle u, b \rangle \leq \langle t, a \rangle$ if $t \subset u$, $a \subset b$ and $a \cap u \setminus t = \emptyset$. We will refer to the union of the first coordinates of conditions in the generic filter as the generic subset of ω , and denote it by \dot{a}_{gen} .

Example 2.8. Let J be an ideal on ω . The forcing $P(J)$ has the continuous reading of names if and only if J is a P-ideal.

Proof. Let I be the σ -ideal on 2^ω associated with the forcing $P(J)$, namely I is the collection of those sets $B \subset 2^\omega$ for which it is outright forced that $\chi(\dot{a}_{gen}) \notin B$. Thus the poset P_I is in the forcing sense equivalent to the poset $P(J)$, with a canonical correspondence between the respective generic objects.

First suppose that J is not a P -ideal, as witnessed by a countable collection $\{a_n : n \in \omega\}$ of sets in the ideal such that no set in the ideal contains each of them modulo a finite set. Consider the Borel function $f : 2^\omega \rightarrow 2^\omega$ defined by $f(r)(n) = \text{the parity of the size of the set } \{m \in a_n : g(m) = 1\}$. The function f is defined on an I -large set, and we claim that it cannot be reduced to a continuous function on an I -positive Borel set.

Suppose that B is an I -positive Borel set, and $\langle t, b \rangle \Vdash \chi(\dot{a}_{gen}) \in \dot{B}$. Thinning out the set B we may assume that it consists only of functions r such that $\forall m \in \omega (t(m) = 1 \rightarrow r(m) = 1 \text{ and } m \in b \rightarrow r(m) = 0)$. Let n be such that the set $a_n \setminus b$ is infinite. It is not difficult to see that both sets $\{r \in B : f(r)(n) = 0\}$ and $\{r \in B : f(r)(n) = 1\}$ are dense in the set B , and therefore the function f cannot be continuous on B .

Now suppose that J is a P -ideal, $B \notin I$ is a Borel set, and $f : B \rightarrow \omega^\omega$ is a Borel function. Let M be a countable elementary submodel of a large enough structure, let $a \subset \omega$ be a set in the ideal J which modulo finite contains all sets in $J \cap M$, and for every number n consider the sets $C_n = \{r \in B : r \text{ is } M\text{-generic and for all } k > n, r(k) = 1 \rightarrow k \notin a\}$. Since the poset P_I is c.c.c., the set $B \setminus \bigcup_n C_n$ is in the ideal I and there must be a number $n \in \omega$ such that the Borel set C_n is I -positive. Set $C = C_n \subset B$; we will be done if we show that the function $f \upharpoonright C$ is continuous.

Suppose $r \in C$ and $O \subset \omega^\omega$ is a basic open set such that $f(r) \in O$. We must produce a basic open set $P \subset 2^\omega$ such that $r \in P$ and for every real $s \in P \cap C$, $f(r) \in O$. Look at the M -generic filter $G \subset M \cap P(J)$ associated with the real r : $G = \{\langle t, b \rangle \in P(J) \cap M : \forall m \in \omega m \in t \rightarrow r(m) = 1 \wedge m \in b \rightarrow r(m) = 0\}$. By the forcing theorem, there must be a condition $\langle t, b \rangle \in G$ such that it forces $f(\chi(\dot{a}_{gen})) \in O$. Let $m \in \omega$ be a natural number larger than n , larger than all elements of the finite set t , and larger than all elements of the finite set $b \setminus c$. It is enough to show that whenever $s \in C$ is a real such that $s \upharpoonright m = r \upharpoonright m$ then $f(s) \in O$. A brief inspection reveals that the condition $\langle t, b \rangle$ belongs to the M -generic ultrafilter associated with the real s , and by the forcing theorem applied in the model M , it must be the case that $f(s) \in O$ as desired. \square

A similar proof can be used to show that the Hechler forcing in the natural presentation has the continuous reading of names, while the eventually different real forcing does not have the continuous reading of names.

Example 2.9. The *eventually different real forcing* [2] 7.4.8 does not have the continuous reading of names. The forcing is the set of all pairs $\langle t, x \rangle$ where $t \in \omega^{<\omega}$ and $x \subset \omega^\omega$ is a finite set, and $\langle u, y \rangle \subset \langle t, x \rangle$ if $t \subset u$, $x \subset y$, and $\bigcup x \cap u \setminus t = \emptyset$. The generic function is the union of the first coordinates of the conditions of the generic filter, and it is eventually different from all functions in ω^ω in the ground model. Let I be the σ -ideal associated with this forcing.

As in the previous proof, the function $f(r)(n)$ = the parity of the size of the set $\{m \in \omega : r(m) = n\}$ is a function which cannot be reduced to a continuous function on an I -positive Borel set.

The continuous reading of names is a rather slippery property of ideals. It is not preserved under Borel isomorphism of ideals. This is to say that there are σ -ideals I and J on Polish spaces X and Y and a Borel bijection $f : X \rightarrow Y$ such that a set $A \subset Y$ is in the ideal J iff its f -preimage is in the ideal I , but the poset P_I does have the continuous reading of names while P_J does not. An instructive example is that of the Steprāns forcing, 2.6. Note that since Borel injective images of Borel sets are Borel, in this case the function f can be naturally extended to an isomorphism of the posets P_I and P_J . This means that the continuous reading of names is, in fact, a property of a presentation of forcing as opposed to a property of the forcing itself. Even so, the continuous reading of names is perceived as a natural and useful property. We state two of its important features.

Claim 2.10. *(LC) If I, J are universally Baire σ -ideals on Polish spaces such that the forcings P_I, P_J are provably proper and have the continuous reading of names then the iteration $P_I * P_J$ in the natural presentation has the continuous reading of names.*

Proof. For simplicity assume that the Polish spaces in question are both just 2^ω . To obtain the natural presentation of the iteration, use the work of [16]. Add all the universally Baire sets without Borel I -positive (or J -positive, respectively) subsets to the ideal I or J respectively. Consider the Fubini product $I * J$ of the two ideals on the space $2^\omega \times 2^\omega$. Then $P_I * P_J$ is in the forcing sense naturally isomorphic to $P_{I * J}$. Moreover, every $I * J$ -positive Borel set contains a *good* Borel subset, that is a set B such that its projection $p(B)$ into the first coordinate is Borel and for every real $r \in p(B)$ the vertical section B_r is J -positive.

So let $B \subset 2^\omega \times 2^\omega$ be a good $I * J$ -positive Borel set and $f : B \rightarrow 2^\omega$ be a Borel function. For every real $r \in p(B)$ there is a J -positive Borel subset of the vertical section B_r on which the function f is continuous. Using universally Baire selection and [16] 2.2.1(2), there is a good $I * J$ -positive set $C \subset B$ such that for every real $r \in p(C)$, the function f is continuous on the vertical section C_r and moreover the function $g : p(C) \rightarrow \mathcal{P}(2^{<\omega} \times 2^{<\omega})$ is Borel, where $\langle u, v \rangle \in g(r)$ if for every real $s \in C_r$ extending u the functional value $f(r, s)$ extends v . Let $D \subset p(C)$ be a Borel I -positive set on which the function g is continuous. Let $E \subset C$ be the set $\{\langle r, s \rangle \in C : r \in D\}$. It is not difficult to verify that the set E is Borel, $I * J$ -positive, and the function $f \upharpoonright E$ is continuous. \square

We do not know if the continuous reading of names is preserved under (countably transfinite) (countable support) iterations of (definable) proper forcing notions, even though in all natural particular cases this can be manually checked to hold true.

Claim 2.11. [16] 2.2.2(2) Suppose that I is a σ -ideal on a Polish space. If P_I is a proper forcing notion with the continuous reading of names then every Borel I -positive set has a G_δ I -positive subset.

Proof. For simplicity assume that the underlying space is 2^ω . Suppose that B is an I -positive Borel set and T is a tree on $2 \times \omega$ projecting to it. Since the poset P_I is proper, there must be an I -positive Borel set $D \subset B$ and a Borel function $f : D \rightarrow [T]$ such that for every element $r \in D$, the first coordinate of the branch $f(r)$ of the tree is r itself [16] Lemma 2.2.1(2). By the continuous reading of names we may assume that the function f is continuous on D . Every partial continuous function can be extended to a continuous function with a G_δ domain. Let $D \subset C$, $f \subset g$ be such a G_δ set and a continuous extension, with D still dense in C . It is immediate that $g : C \rightarrow [T]$ is a function such that for every $r \in C$ the first coordinate of the branch $g(r)$ of the tree T is r itself. Then $C \subset B$ is an I -positive G_δ subset of the set B . \square

The opposite implication does not hold: compact sets are dense in the natural presentation of Steprans forcing [16] 2.3.46, while the continuous reading of names fails.

3 Proof of Theorem 1.2

We will deal with a σ -ideal I on the space 2^ω , the ω^ω case being analogous.

Definition 3.1. The function $\pi : \mathcal{P}(2^{<\omega}) \rightarrow \mathcal{P}(2^\omega)$ is defined by $\pi(a) = \{r \in 2^\omega : \exists^\infty n \ r \restriction n \in a\}$.

Clearly, the range of the function π is exactly the collection of all G_δ subsets of 2^ω , and the function π preserves inclusion. Moreover, if I is a σ -ideal on 2^ω then $a \in \text{tr}(I)$ if and only if $\pi(a) \in I$, and the map $\pi \restriction Q_{\text{tr}(I)} : Q_{\text{tr}(I)} \rightarrow P_I$ preserves compatibility. For the remainder of the section fix a σ -ideal I on 2^ω such that the poset P_I is proper and has the continuous reading of names, and write $J = \text{tr}(I)$.

Claim 3.2. 1. For every set $a \notin Q_J$ and for every I -positive G_δ subset $B \subset \pi(a)$ there is a set $b \subset a$ such that $\pi(b) = B$.

2. Q_J forces $\pi''\dot{G}$ to be a P_I -generic filter, where \dot{G} is the name for the Q_J generic filter.

Proof. The second item immediately follows from the first. For the first one, suppose $B \subset \pi(a)$ is an I -positive G_δ set, $B = \bigcap_n O_n$ for some open sets O_n . By induction on $n \in \omega$ build sets $a_n \subset a$ in the following way:

1. Each $a_n \subset 2^{<\omega}$ is an antichain and it refines a_{n-1} . For notational convenience let $a_{-1} = \{0\}$.
2. $B \subset \bigcup_{t \in a_n} [t] \subset O_n$.

After the construction is complete, clearly writing $b = \bigcup_n a_n$ we will have $\pi(b) = B$ as required. Suppose the antichain a_n has been obtained. For each $t \in a_n$ let $c(t)$ be the collection of all proper extensions $u \in 2^{<\omega}$ of t such that the basic open set determined by u is a subset of O_{n+1} , and no proper initial segment of u longer than t has this property. Note that $c(t) \subset 2^{<\omega}$ is an antichain. For each $u \in c(t)$ let $d(u)$ be the collection of all proper extensions v of u which are in the set a , such that no proper initial segment of v longer than u has this property. Note that each $d(u)$ is an antichain. It is not difficult to verify that the set $a_{n+1} = \bigcup_u d(u)$ has the desired properties. \square

Claim 3.3. *The poset Q_J is proper.*

Proof. Let M be a countable elementary submodel of a large structure and let $a \in Q_J \cap M$ be a condition. Let $\langle D_n : n \in \omega \rangle$ be an enumeration of all open dense subsets of the poset Q_J in the model M . We will find sets $a_n \subset a$ and functions $g_n : a_n \rightarrow D_n \cap M$ so that

1. each set $a_n \subset 2^{<\omega}$ is an antichain and it refines a_{n-1} . For notational convenience let $a_{-1} = \{0\}$.
2. the set $b = \bigcup_n a_n \subset a$ is J -positive
3. for each sequence $t \in a_n$ the set $\{u \in b : t \subset u\}$ is a subset of $g_n(t)$.

It immediately follows that the set $b \subset a$ is the required M -master condition in the poset Q_J . To see this, choose a J -positive set $c \subset b$ and an open dense set $D = D_n \in M$ for some number n . For each sequence $t \in a_n$ write $b_t = \{u \in b : t \subset u\}$. Since the set $a_n \subset 2^{<\omega}$ is an antichain, it is the case that $\pi(c) = \bigcup_{t \in a_n} \pi(c \cap b_t)$ and therefore one of the sets $c \cap b_t : t \in a_n$ must be J -positive. Such set $c \cap b_t \subset c$ has the condition $g_n(t) \in D_n \cap M$ above it as required.

To perform the construction, find an M -master condition $B \subset \pi(a)$ for the poset P_I . Thinning out the condition B we may assume that for every dense set $E \in M$ of the poset P_I , $B \subset \bigcup(E \cap M)$. Thinning out the condition B even further, by Claim 2.2, we may assume that for every set $C \in P_I \cap M$ the intersection $C \cap B$ is relatively open in B . Thinning out the condition B further we may assume that it is a G_δ set such that for every basic open set O , $B \cap O \notin I \leftrightarrow B \cap O \neq \emptyset$. Fix a representation $B = \bigcap_n O_n$, for some open sets O_n .

The induction hypotheses on the construction of the sets a_n are the following.

1. Each $a_n \subset 2^{<\omega}$ is an antichain and it refines a_{n-1} . For notational convenience let $a_{-1} = \{0\}$.
2. $B \subset \bigcup_{t \in a_n} [t] \subset O_n$
3. For every n , $g_n(t) \subset \{u \in a : t \subset u\}$ is a condition in the open dense set D_n . For every $n \in m$, $t \in a_n$ and $u \in a_m$ such that $t \subset u$, $u \in g_n(t)$ and $g_m(u) \subset g_n(t)$. For notational convenience put $g_{-1}(0) = a$.

4. For each $t \in a_n$ it is the case $B \cap [t]$ is a nonempty subset of $\pi(g_n(t))$.

Now suppose that a_n, g_n have been constructed. Fix a node $t \in a_n$. We will show how the part of the antichain a_{n+1} below t will be constructed. Let E be the part of the open dense set $D_{n+1} \subset Q_J$ below the condition $g_n(t) \in Q_J$. Claim 3.2 shows that the set $\pi''(E)$ is dense below the condition $\pi(g_n(t))$. Then $B \cap [t] \subset \bigcup (\pi''(E) \cap M) = \bigcup \pi''(E \cap M)$ by the choice of the M -master condition B . Note that for every condition $p \in E \cap M$ the set $\pi(p) \cap B \subset B \cap [t]$ is relatively open by the choice of the condition $B \in P_I$ again. It is now easy to build an antichain $b \subset 2^{<\omega}$ below the node t so that for every $u \in b$ it is the case that $[u] \subset O_n$ and there is a condition $p(u) \in E \cap M$ such that $B \cap [u]$ is a nonempty subset of $\pi(p(u))$, and $B \cap [t] \subset \bigcup_{u \in b} [u]$. Let c be then the collection of all nodes $v \in 2^{<\omega}$ such that there is some $u \in b$ such that $u \subset v$, $v \in p(u)$, $B \cap [v] \neq \emptyset$ and no proper initial segment of v is an extension of u in $p(u)$. The set c is an antichain below the node t , and it is the part of the antichain a_{n+1} below t . For every node $v \in c$ let $g_{n+1}(v) = p(u) \cap \{w \in 2^{<\omega} : v \subset w\}$. The induction hypotheses are easily seen to be satisfied. \square

Claim 3.4. *The remainder poset $R = Q_J/P_I$ preserves stationary subsets of ω_1 and it is \aleph_0 -distributive.*

Proof. Here the remainder poset R is computed via the Q_J -name for the P_I -generic filter obtained in Claim 3.2. Note that writing \dot{r}_{gen} for the canonical P_I -generic real we have $t \subset \dot{r}_{gen} \leftrightarrow$ the set $\{u \in 2^{<\omega} : t \subset u\}$ is in the Q_J -generic filter, this for every sequence $t \in 2^{<\omega}$.

The fact that $P_I \Vdash \dot{R}$ is stationary preserving" follows abstractly from the proof of the previous claim: if M is a countable elementary submodel and $B \in P_I$ is any M -master condition for the poset P_I then there is an M -master condition $b \in Q_J$ such that $\pi(b) \subset B$. For the distributivity, suppose that \dot{f} is a Q_J -name for an ω -sequence of ordinals. We must prove that $\dot{f} \in V[\dot{r}_{gen}]$.

To this end, revisit the proof of the previous claim again. Assume that $\dot{f} \in M$ and for each number $k \in \omega$ find a number $n_k \in \omega$ such that the conditions in the open dense set $D_{n_k} \subset Q_J$ decide the value of $\dot{f}(\check{k})$. Look again at the master condition $b = \bigcup_n a_n$. It is not difficult to see that b forces that for each $n \in \omega$ there is exactly one initial segment of the real \dot{r}_{gen} in the set a_n ; call it t_n . Consequently, the sequence \dot{f} can be recovered in the model $V[\dot{r}_{gen}]$ by the following formula: $\dot{f}(\check{k})$ is that ordinal which is forced by the condition $g_{n_k}(t_{n_k})$ to be the value of $\dot{f}(\check{k})$. \square

It is interesting to see what the \aleph_0 -distributive tail Q_J/P_I can be. From the definitions it is equal to the collection of all ground model sets $a \subset 2^{<\omega}$ such that the P_I -generic real has infinitely many initial segments in a , ordered by inclusion. In many cases it is, in the forcing sense, equivalent to $\mathcal{P}(\omega)/\text{fin}$ of the P_I extension. To prove this it is just enough to show that P_I forces every infinite subset of the generic real \dot{r}_{gen} (understood now as a path through $2^{<\omega}$ or $\omega^{<\omega}$) to have an infinite subset of the form $a \cap \dot{r}_{gen}$ for some set a in the ground

model. We can verify this property in a great number of cases and disprove in others, but we do not have a suitable general criterion.

Proposition 3.5. *Let I be a σ -ideal on 2^ω σ -generated by a σ -compact family of closed sets. The forcing P_I is proper and bounding, and writing $J = \text{tr}(I)$, $Q_J = P_I * \mathcal{P}(\omega)/\text{fin}$.*

Proof. The ideals I considered in this proposition form a class considered in [5]. There it is proved that the poset P_I is proper and bounding. In fact a standard determinacy argument [5] shows the following. Fix a σ -ideal I σ -generated by a σ -compact collection $F = \bigcup_n F_n$ of closed sets, where the sets $F_n \subset K(2^\omega)$ are closed. Call a tree $T \subset 2^{<\omega}$ *I-good* if for every node $t \in T$ and every number n there is a number m such that no set in F_n meets all the open sets determined by the extensions of the node t in the tree T of length m . Then a Borel set $B \subset 2^\omega$ is *I-positive* if and only if it contains all branches of some *I-good* tree. Therefore the poset of *I-good* trees is naturally isomorphic with a dense subset of the poset P_I and below we will identify it with P_I .

We will show that $P_I \Vdash$ “every infinite subset of \dot{r}_{gen} has an infinite subset of the form $a \cap \dot{r}_{gen}$ for some set a in the ground model”. Suppose $T \in P_I$ is an *I-good* tree, $T \Vdash$ “ $\dot{x} \subset \dot{r}_{gen}$ is an infinite set”. A standard fusion argument will give an *I-good* tree $S \subset T$ such that for every number n there is $m > n$ such that every sequence $s \in S$ of length m has an initial segment of length $\geq n$ in the set $a = \{t \in S : S \restriction t \Vdash \dot{t} \in \dot{x}\} \subset S$. Clearly $S \Vdash$ “ $\dot{a} \cap \dot{r}_{gen}$ is an infinite subset of \dot{x} ” as desired. \square

The following definition is not standard. It is an attempt to restate a commonly used combinatorial forcing property in topological terms.

Definition 3.6. Let I be a σ -ideal on some Polish space X with a fixed metric d . We say that the poset P_I has the *pure decision property* (with respect to the metric d) if for every *I-positive* Borel set $B \subset X$ and every Borel map $f : (B, d) \rightarrow (Y, e)$ into a compact metric space there is a Borel *I-positive* set $C \subset B$ on which the map f is a contraction.

Example 3.7. The Laver forcing has the pure decision property in the standard representation, with respect to the metric of least difference on ω^ω : $d(x, y) = 2^{-n}$ where n is the smallest number where the functions $x, y \in \omega^\omega$ differ.

Proof. Let B be Borel *I-positive* set and $f : (B, d) \rightarrow (Y, e)$ be a Borel map into a compact metric space. Thinning out the set B if necessary we may assume that $B = [T]$ for some Laver tree $T \subset \omega^{<\omega}$. To simplify the notation assume that T has an empty trunk.

Before we proceed recall the well known fact that for every Laver tree S and Borel partition $[S] = \bigcup_{i \in n} A_i$ into finitely many pieces there is a Laver tree $U \subset S$ with the same trunk such that the set $[U]$ is included in one of the pieces of the partition.

Now for every n find a finite 2^{-n-1} -network $y_n \subset Y$, that is, a set such that every point of the space Y is 2^{-n-1} -close to one of its elements. By induction

on $n \in \omega$ build a fusion sequence of Laver trees T_n so that $T_0 = T$, T_{n+1} agrees with T_n on sequences of length $n+1$ and for every such a sequence $t \in T_n$ there is an element $x_t \in y_n$ such that for every path r through T_{n+1} extending the sequence t , the element $f(r) \in Y$ is 2^{-n-1} -close to x_t . This is possible by the observation in the previous paragraph. Note that by the triangle inequality this means that for two such paths r_0, r_1 the elements $f(r_0), f(r_1) \in Y$ will have e -distance $\leq 2^{-n}$. Let S be the fusion of the sequence of trees T_n . It is not difficult to see that the set $C = [S]$ has the required properties. \square

Proposition 3.8. *If I is a σ -ideal on ω^ω such that the poset P_I is proper and has the pure decision property with respect to the metric of least difference on ω^ω . Then $Q_{tr(I)} = P_I * \mathcal{P}(\omega)/\text{fin}$.*

Proof. Note that the pure decision property implies the continuous reading of names.

Suppose that $B \in P_I$ forces $\dot{x} \subset \dot{r}_{gen}$ to be an infinite set. Since the poset P_I is proper, thinning out the set B if necessary we can find a Borel map $f : B \rightarrow 2^\omega$ such that $B \Vdash \dot{x} = \{\dot{r}_{gen} \restriction n : n \in \dot{f}(\dot{r}_{gen})\}$. Consider the metric e of least difference on 2^ω and use the pure decision property to find an I -positive set $C \subset B$ such that $f : C \rightarrow 2^\omega$ is a contraction. This means that for every sequence $t \in \omega^{<\omega}$, all reals $r \in C$ extending the sequence t return the same value $b(t) \in 2$ for $f(r)(|t|)$. Let $a = \{t \in \omega^{<\omega} : b(t) = 1\}$. It follows from the definitions that $C \Vdash \dot{x} = a \cap \dot{r}_{gen}$, and the lemma follows. \square

Example 3.9. The Cohen poset forces that there is an infinite set $x \subset \dot{r}_{gen}$ without an infinite subset of the form $a \cap \dot{r}_{gen}$, $a \in V$. Just let an initial segment t of \dot{r}_{gen} into \dot{x} if and only if $\dot{r}_{gen}(|t|) = 0$.

As a final remark in this section, once we produced so many ideals J for which the factor forcing Q_J is proper, we should also produce some for which it is not proper. The following Proposition of independent interest shows how to do exactly that in several ways. First, an instrumental definition.

Definition 3.10. Let β be a limit ordinal. We say that an inclusion-decreasing sequence $\langle I_\alpha : \alpha \in \beta \rangle$ of σ -ideals on a Polish space *does not stabilize* if for every ordinal $\alpha \in \beta$ and every I_α -positive Borel set B there is an ordinal $\alpha \in \gamma \in \beta$ and a Borel set $C \subset B$ which is I_α -small but I_γ -positive. This is equivalent to saying that, writing $I = \bigcap_\alpha I_\alpha$, the sets $I_\alpha \cap P_I$ are all dense in P_I . Restated again, this is equivalent to saying that for every I -positive Borel set B , $I \restriction B \neq I_\alpha \restriction B$ —hence the terminology.

Similarly, we say that an inclusion-decreasing sequence $\langle J_\alpha : \alpha \in \beta \rangle$ of ideals on some countable set X does not stabilize if for every ordinal $\alpha \in \beta$ and J_α -positive set $a \subset X$ there is an ordinal $\alpha \in \gamma \in \beta$ and a set $b \subset a$ which is J_α -small but J_γ -positive. This is the same as to say, writing $J = \bigcap_\alpha J_\alpha$, that the set $J_\alpha \cap Q_J$ are dense in the factor forcing Q_J .

Proposition 3.11. *Assume the Continuum Hypothesis. If I is a σ -ideal on a Polish space, then*

1. P_I collapses \aleph_1 if and only if $I = \bigcap_{n \in \omega} I_n$ for an inclusion-decreasing sequence of σ -ideals which does not stabilize.
 2. Suppose P_I preserves \aleph_1 . P_I is nowhere c.c.c. if and only if $I = \bigcap_{\alpha \in \omega_1} I_\alpha$ for an inclusion-decreasing sequence of σ -ideals which does not stabilize.
- If J is an ideal on a countable set, then
3. Q_J adds an unbounded real if and only if $J = \bigcap_{n \in \omega} J_n$ for an inclusion-decreasing sequence of ideals which does not stabilize.
 4. If $J = \bigcap_{n \in \omega} J_n$ for an inclusion-decreasing sequence of P -ideals which does not stabilize, then Q_J collapses \aleph_1 .

Proof. For the first equivalence, assume that P_I collapses \aleph_1 . Let $\dot{f} : \omega \rightarrow \omega_1$ be a name for a function with cofinal range. For every number $n \in \omega$ let I_n be the ideal generated by sets $B \in P_I$ which force the first n values of the function \dot{f} to be bounded by some fixed countable ordinal, together with all sets in the ideal I . It is clear that $\langle I_n : n \in \omega \rangle$ is an inclusion-decreasing sequence of σ -ideals which does not stabilize, and $I = \bigcap_n I_n$. On the other hand, suppose that $I = \bigcap_n I_n$ for some inclusion-decreasing sequence of σ -ideals which does not stabilize. Since the ideals I_n are dense in the poset P_I , we can pick a maximal antichain $A_n \subset I_n$ from each, and by CH it will be enough to show that every condition in P_I is compatible with uncountably many elements of one of these antichains. Indeed, if $B \in P_I$ is a condition, then $B \notin I_n$ for some number n , and B must be compatible with uncountably many elements of the antichain A_n , because if $X \subset A_n$ is a countable set, then $C = \bigcup X \in I_n$ and the condition $B \setminus C \notin I_n$ is a condition incompatible with all elements of the set X .

For the second equivalence, assume that the poset P_I preserves \aleph_1 and is nowhere c.c.c. Then there is a name \dot{f} for a function from ω_1 to itself which is not bounded by any ground model such function. To see this, let $\langle M_\alpha : \alpha \in \omega_1 \rangle$ be a tower of countable elementary submodels of some large structure, and define $\dot{f}(\alpha) = \min\{\beta \in \omega_1 : \text{for every maximal antichain } A \in M_\alpha \text{ the unique element in it which belongs to the generic filter is in the model } M_\beta\}$. Since the forcing P_I preserves \aleph_1 , and by CH $P_I \subset \bigcup_\alpha M_\alpha$, this is well-defined. If $p \in P_I$ is a condition and $g : \omega_1 \rightarrow \omega_1$ is a function, find an ordinal $\alpha \in \omega_1$ such that $p \in M_\alpha$, a maximal antichain $A \in M_\alpha$ which has uncountably many elements below the condition p , and an element $q \in A \setminus M_{g(\alpha)}$ below the condition p . Then $q \Vdash "g(\alpha) \in \dot{f}(\alpha)"$, and it follows that the function \dot{f} is unbounded.

Now, given an ordinal α let I_α be the ideal generated by the sets $B \in P_I$ for which there is a countable ordinal β such that B force all values $\{\dot{f}(\gamma) : \gamma \in \alpha\}$ to be smaller than β , together with all sets in the ideal I . It is clear that $\langle I_\alpha : \alpha \in \omega_1 \rangle$ is an inclusion decreasing sequence of σ -ideals. Since the function \dot{f} is not dominated by any ground model function, it is the case that $I = \bigcap_\alpha I_\alpha$, and since the forcing P_I preserves \aleph_1 , the sequence of ideals does not stabilize.

For the other direction, let $I = \bigcap_\alpha I_\alpha$. Suppose $B \in P_I$ is a Borel set; we must find an uncountable antichain below it. It must be the case that $B \notin I_\alpha$ for

some countable ordinal α . Now since the sequence of ideals does not stabilize, the ideal I_α is dense in the poset P_I , and therefore there must be a maximal antichain A below B which consists solely of I_α -small sets. This antichain must be uncountable, because otherwise $\bigcup A \in I_\alpha$ and $B \setminus \bigcup A \notin I_\alpha$ is a condition in P_I which avoids all elements of the maximal antichain, a contradiction.

For the third equivalence, first suppose that $Q_J \Vdash \dot{f} \in \omega^\omega$ is an unbounded function". Let $J_n = \{a \subset \omega : \text{there is a number } m \text{ forces the first } n \text{ values of the function } \dot{f} \text{ to be smaller than } m\}$. It is immediate that $\langle J_n : n \in \omega \rangle$ is an inclusion-decreasing sequence of ideals which does not stabilize. Since \dot{f} is forced unbounded, $J = \bigcap_n J_n$. On the other hand, suppose that $J = \bigcap_n J_n$ is the intersection of an inclusion decreasing sequence of ideals which does not stabilize. Each ideal J_n is dense in Q_J , so we can find a maximal antichain $A_n \subset J_n$. Now suppose $a \in Q_J$. Either there is no condition $b \subset a$ which is compatible with at most countably many elements of $\bigcup_n A_n$. In such a case $a \Vdash \aleph_1$ is collapsed and by the CH an unbounded real is added". Or there is such a condition b , compatible only with elements $\{a_n^k : k \in \omega\}$ of the antichain A_n . Let $\dot{f} \in \omega^\omega$ be defined by $f(n) = \text{the unique } k \text{ such that } a_n^k \text{ is in the generic filter}$. The condition b forces this function to be well-defined, and we will be done if we prove that it forces it not to be bounded by any ground model function. Indeed, if $c \subset b$ is a condition and $g \in \omega^\omega$ is a function, it must be the case that $c \notin J_n$ for some number n , $d = \bigcup_{k \in g(n)} a_n^k \in J_n$, $c \setminus d \notin J_n$ and clearly $c \setminus d \subset c$ is a condition forcing $g(n) \leq \dot{f}(n)$.

Finally, for the fourth item, suppose that $J = \bigcap_n J_n$ is an intersection of an inclusion decreasing sequence of P-ideals which does not stabilize. Every ideal J_n is dense in the factor Q_J , therefore we can find a maximal antichain $A_n \subset J_n$. We will be done if we show that every condition $a \in Q_J$ is compatible with uncountably many elements of one of these antichains. Indeed, if $a \in Q_J$, then $a \notin J_n$ for some number n , and a must be compatible with uncountably many elements of the antichain A_n . This is true because if $X \subset A_n$ is a countable set then there is a set $b \in J_n$ containing all elements of X modulo finite, and then $a \setminus b \notin J_n$ is a condition which is incompatible with all elements of the set X ! \square

The following example answers a question of Ilijas Farah–Question 4.3 of [4].

Example 3.12. An analytic P-ideal J such that the factor Q_J collapses \aleph_1 . Let $\langle \alpha_n : n \in \omega \rangle$ be a decreasing sequence of positive real numbers smaller than 1. Let J_n be the summable P-ideal associated with the weight function $k \mapsto k^{-\alpha_n}$. We claim that $\langle J_n : n \in \omega \rangle$ is an inclusion-decreasing sequence of ideals which does not stabilize. The inclusions are clear. To see that stabilization is impossible, choose a number n and a set $a \notin J_n$. We will produce a set $b \subset a$, $b \in J_n \setminus J_{n+1}$. By induction on $m \in \omega$ find mutually disjoint finite sets $b_m \subset a$ such that $\sum_{k \in b_m} k^{-\alpha_n} \leq 2^{-m}$ while $\sum_{k \in b_m} k^{-\alpha_{n+1}} \geq 1$. Then $b = \bigcup_m b_m$ will be as desired. To find the set b_m , first find a number $k_m \in \omega$ such that for every $k > k_m$ it is the case that $k^{-\alpha_n} \leq 2^{-m-1} k^{-\alpha_{n+1}} \leq 2^{-m-1}$ and then find a finite set b_m consisting of numbers larger than k_m such that the sum $\sum_{k \in b_m} k^{-\alpha_n}$ is

between 2^{-m-1} and 2^{-m} .

Now let $J = \bigcap_n J_n$. This is an $F_{\sigma\delta}$ ideal, and a simple diagonalization arguments show that it is a tall P-ideal. The Proposition shows that the factor Q_J collapses \aleph_1 in the presence of CH. If CH fails, the argument only shows that Q_J is not proper, and we do not know if it has to collapse \mathfrak{c} to \aleph_0 .

Note that this ideal is of minimal possible complexity for the factor Q_J to be improper. All factors of F_σ forcings are σ -closed by a theorem of Ilijas Farah.

Example 3.13. Let $\langle K_n : n \in \omega \rangle$ be a decreasing sequence of ideals on ω which does not stabilize. Consider the forcings L_n of all trees $T \subset \omega^{<\omega}$ such that every node $s \in T$ longer than some fixed $t \in T$ splits into K_n -positively many immediate successors. It is not difficult to show that the posets L_n are proper and have the continuous reading of names—the arguments closely follow those for Laver forcing. Let $I_n : n \in \omega$ be the ideals on ω^ω associated with these forcings; a Borel set $B \subset \omega^\omega$ is I_n -positive if and only if $[T] \subset B$ for some tree $T \in L_n$. It is not difficult to see that the ideals form an inclusion-decreasing sequence which does not stabilize. Let $J_n = \text{tr}(I_n)$, let $I = \bigcap_n I_n$, and let $J = \bigcap_n J_n = \text{tr}(I)$. Since P_I regularly embeds into Q_J by Claim 3.2 and P_I is not proper by the Proposition, it must be the case that Q_J is not proper either.

4 The complexity of the trace ideals

There are a number of questions concerning the structure of the factor posets Q_J for various simply definable ideals J on ω . In order to address them directly, we must analyze the complexity of the trace ideals.

Proposition 4.1. *Suppose that I is a σ -ideal on 2^ω such that the factor forcing P_I is proper and has the continuous reading of names, and every analytic I -positive set has a Borel I -positive subset. The following are equivalent:*

- I is Π_1^1 on Σ_1^1 —[7] 29.E, 35.9.
- the trace ideal $\text{tr}(I)$ is co-analytic.

Proof. The top to bottom direction follows immediately from the definitions. For the bottom to top direction suppose that $\text{tr}(I)$ is co-analytic. Let $B \subset 2^\omega$ be an analytic set, with a tree T such that $B = p[T]$. The proof will be complete if we show that $B \notin I$ is equivalent to the following analytic statement: there is a set $b \subset 2^{<\omega}$ decomposed into antichains $b = \bigcup_n a_n$ and a function $g : b \rightarrow T$ for $n \in \omega$ such that

- the antichain a_{n+1} refines a_n
- g preserves extension and whenever $u \in a_n$ then $g(u)$ is a sequence of length n whose first coordinates form an initial segment of u
- $b \notin \text{tr}(I)$.

First, if there are such objects b and g , it is clear that the I -positive set $\pi(b)$ is a subset of B and therefore $B \notin I$. On the other hand, if $B \notin I$ then B has a Borel I -positive subset C by the assumptions, and by the properness and the continuous reading of names of the poset P_I there even is a Borel I -positive G_δ set $D \subset C$ and a continuous function $f : D \rightarrow [T]$ such that for every real $r \in D$, the first coordinate of the value $f(r)$ is just r itself. It is then easy to construct b and g as above in such a way that $\pi(b) = D$ and for every real $r \in D$, $f(r) = \bigcup_{u \subset r} g(u)$. \square

This lemma gives us a rather good criterion for checking whether a given trace ideal is co-analytic or not. If the poset P_I adds a dominating real then the ideal I is not Π_1^1 on Σ_1^1 , [16], C.0.16. A quick review of forcings used in practice shows that many of them which do not add dominating reals (such as the Cohen or Solovay real) are associated with Π_1^1 on Σ_1^1 ideals. However, Arnold Miller [10] constructed a definable c.c.c. ideal I such that the poset P_I does not add a dominating real while the ideal I is still not Π_1^1 on Σ_1^1 . Therefore a careful check for this property is frequently necessary.

Example 4.2. Suppose that J is an ideal on ω containing all finite sets, and let I be the σ -ideal on 2^ω associated with the Prikry forcing $P(J)$ from Example 2.8. The ideal I is Π_1^1 on Σ_1^1 if and only if the ideal J is F_σ .

Proof. For the right-to-left direction note that the ideal I is ergodic in the sense of [16]: for every Borel set $B \subset 2^\omega$ closed under finite changes, either $B \in I$ or $2^\omega \setminus B \in I$. Now if I is Π_1^1 on Σ_1^1 then the restriction of the trace $tr(I)$ to trees on $2^{<\omega}$ is analytic: $T \in tr(I)$ if and only if $[T] \in I$ if and only if $2^\omega \setminus$ the closure of the set $[T]$ under finite changes is I -positive, which is an analytic statement. Analytic ideals of compact sets are G_δ by a theorem of Kechris, Louveau, and Woodin [8]. Now it is not difficult to see that for a set $a \subset \omega$, $a \in J$ iff $T_a \notin tr(I)$, where T_a is the set of all sequences $t \in 2^{<\omega}$ for which $\forall n \ t(n) = 1 \leftrightarrow n \notin a$. However, $T_a \notin tr(I)$ is an F_σ condition!

For the left-to-right direction, write $J^{<\omega} = \{a \subset [\omega]^{<\aleph_0} \setminus \{0\} : \exists b \in J \forall x \in a \ x \cap b \neq \emptyset\}$. It is clear that this is an ideal on the set $[\omega]^{<\aleph_0} \setminus \{0\}$. A useful observation:

Claim 4.3. *If J is an F_σ ideal then $J^{<\omega}$ is F_σ again.*

Proof. By a theorem of Mazur [9] there is a lower semicontinuous submeasure μ on $\mathcal{P}(\omega)$ such that $J = \{a \subset \omega : \mu(a) < \infty\}$. Let $\mu^{<\omega}$ be a function on $\mathcal{P}([\omega]^{<\aleph_0})$ defined by $\mu^{<\omega}(b) = \inf\{\mu(a) : \forall x \in b \ x \cap a \neq \emptyset\}$. It is not difficult to verify that this is a lower semicontinuous submeasure such that $J^{<\omega} = \{b \subset [\omega]^{<\aleph_0} : \mu^{<\omega}(b) < \infty\}$. The claim follows.

In fact the proof of the Example shows that if the ideal J is not F_σ then the ideal $J^{<\omega}$ is not even analytic. \square

By [16] C.0.14, to prove the left-to-right implication of the Example it is just necessary to show that the collection of countable subsets of $P(J)$ which are maximal antichains is a Borel set. In order to do this, let $A \subset P(J)$ be a

countable set. Then A is a maximal antichain if and only if it is an antichain and for every finite set $t \subset \omega$, every condition of the form $\langle t, a \rangle$ is compatible with some element of A . The latter condition is equivalent to: either there is some condition $\langle u, b \rangle \in A$ such that $u \subset t$ and $b \cap t \setminus u = 0$, or the set $a_t = \{x \subset \omega : \exists b \langle t \cup x, b \rangle \in A\}$ is not in the ideal $J^{<\omega}$. By the Claim, this is a Borel statement. \square

On the other hand, we do not have a good criterion as to when the trace ideal is analytic. We have just a conjecture:

Conjecture 4.4. Suppose that I is a σ -ideal on 2^ω or ω^ω such that the factor poset P_I is proper with continuous reading of names. If the trace ideal is analytic then it is in fact Borel.

This conjecture can be viewed as a variation on the Kechris-Louveau-Woodin theorem [8]. We can verify it in a good number of cases:

Lemma 4.5. *Suppose that I is a σ -ideal on 2^ω such that P_I is proper and bounding. If the trace ideal is analytic then it is Borel.*

Proof. Since the poset P_I is bounding, compact sets are dense in it and it has the continuous reading of names—[16]. If a set $a \subset 2^{<\omega}$ is not in the trace ideal, apply these two properties below the condition $\pi(a)$ to the name $\dot{f}(n)$ = the n -th initial segment of the generic real in the set \check{a} . It follows that a set $a \subset 2^{<\omega}$ is not in the trace ideal if and only if there is a tree T and disjoint finite subsets $\{b_n : n \in \omega\}$ of T such that $[T] \notin I$ and each b_n is a maximal antichain in T consisting only of elements of the set a . What is the complexity of the latter statement? The trace ideal restricted to trees is analytic, therefore Borel by the Kechris-Louveau-Woodin theorem [8], and so this is an analytic statement. Thus the trace ideal is both analytic and co-analytic, therefore Borel. \square

Of course, it is quite interesting to investigate the possibility of the trace ideal being Borel. The classical examples of such behavior are the meager and null ideals. Other examples turn out to be quite hard to find.

Example 4.6. Suppose that J is an analytic P-ideal, and I is the σ -ideal associated with the Prikry poset $P(J)$. Then the following are equivalent:

1. J is F_σ
2. I is Π_1^1 on Σ_1^1
3. $P(J)$ does not add a dominating real
4. the trace ideal is Borel.

Proof. (1) is equivalent with (2) by Example 4.2. (2) implies (3) by [16], C.0.16. (3) implies (1) by a result of Solecki: if an analytic P-ideal J is not F_σ then the ideal $0 \times \text{Fin}$ is Rudin-Blass reducible to J [11], 3.3. Let $f : \omega \rightarrow \omega \times \omega$ be such a finite-to-one reduction. Since the $P(J)$ generic set $\dot{a}_{gen} \subset \omega$ has a finite

intersection with a ground model set $b \subset \omega$ if and only if $b \in J$, it immediately follows that $f''\dot{a}_{gen}$ has a finite intersection with a ground model set $b \subset \omega \times \omega$ if and only if $b \in 0 \times \text{Fin}$. Let $g \in \omega^\omega$ be defined by $g(n) = \min\{m \in \omega : \langle n, m \rangle \in f''\dot{a}_{gen}\}$. A brief inspection reveals that this is a well-defined function modulo finite dominating all ground model functions.

This leaves us with the equivalence of (2) and (4). Note that the forcing P_I has the continuous reading of names by Example 2.8 and so (4) implies (2) by Proposition 4.1. For the opposite direction note that (2) implies the trace ideal is co-analytic by that same Proposition, so it is enough to show from (2) that the trace ideal is analytic. To this end, use the ergodicity of the ideal I again. For a set $a \subset 2^\omega$, $a \in \text{tr}(I)$ if and only if the complement of the closure of the set $\pi(a)$ under finite changes is I -positive, which is an analytic condition by (2). \square

In order to produce many Borel ideals on ω such that their factors are mutually distinct as forcing notions, it is now necessary to show that the posets $P(J)$ are distinct as forcings for different F_σ P -ideals J . This does not seem to be easy but we do have a natural criterion.

Definition 4.7. A forcing P *separates* an ideal K on ω if it introduces a set $x \subset \omega$ which has finite intersection with all K -small sets and infinite intersection with all K -positive sets.

It is natural to suggest that the reason two forcings $P(K)$ and $P(J)$ may be different is because $P(J)$ does not separate K . We will produce a forcing relation free criterion for this situation. First, a couple of basic facts.

Proposition 4.8. *Suppose that K, J are ideals on ω .*

1. *If K is Rudin-Keisler reducible to J and a forcing P separates J then it separates K*
2. *$P(J)$ separates J , $P(J) \times \text{Cohen}$ separates $J^{<\omega}$*
3. *if P separates $J^{<\omega}$ then the poset $P(J)$ regularly embeds into $P \times \text{Cohen}$.*

Proof. Suppose $f \in \omega^\omega$ is a Rudin-Keisler reduction of K to J , and a forcing P separates J by introducing an infinite set $a \subset \omega$. It is straightforward to check that $f''a$ separates K .

For (2), let $a \subset \omega$ be a $P(J)$ -generic set and $f \in \omega^\omega$ be a Cohen real generic over a . A straightforward density argument shows that the set a separates the ideal J . Another density argument will show that the set $b = \{x \subset a : x \text{ is nonempty, finite, and } \max(x) \in f(\min(x))\}$ separates $J^{<\omega}$.

Finally, for (3) assume that $b \subset [\omega]^{<\aleph_0}$ is a set separating the ideal $J^{<\omega}$ and $f \in 2^b$ is a Cohen real generic over b , obtained by the poset R of all finite partial functions from b to 2. We claim that the set $a = \bigcup \{x \in b : f(x) = 1\}$ is $P(J)$ -generic over the ground model. For an ease of notation, if $p \in R$ is a condition, write $t_p = \bigcup \{x \in \text{dom}(p) : p(x) = 1\}$. Now suppose that $D \subset P(J)$

is a ground model open dense set and $p \in R$ is a condition. By a standard genericity argument, it will be enough to find an extension $q \supset p$ such that there is a J -small set $a \in V$ such that the condition $\langle t_q, a \rangle \in P(J)$ is in the open dense set D , and every set in $b \setminus \text{dom}(q)$ has an empty intersection with the set a .

The set $c = \{x \in [\omega]^{<\aleph_0} : \exists a \in J \langle t_p \cup x, a \rangle \in D\} \in V$ is $J^{<\omega}$ -positive, therefore has infinite intersection with the set b . Let $x \in b \cap c \setminus \text{dom}(p)$ be an arbitrary set, and let $a \in J$ be a set such that $\langle t_p \cup x, a \rangle \in D$. Since the set b separates the ideal $J^{<\omega}$, only finitely many of its elements have nonempty intersection with the set a . Let $q \supset p$ be a condition whose domain includes all of these elements and x , and x is the only element of $\text{dom}(p \setminus q)$ on which the function q returns value 1. The condition $q \in R$ is as required. \square

Here is the promised criterion:

Proposition 4.9. *Suppose K is a tall P -ideal and J is an ideal on ω . Then (1) implies (2) implies (3), where*

1. $P(J)$ separates K
2. K is Rudin-Keisler reducible to the ideal $J^{<\omega} \upharpoonright b$ where $b \subset [\omega]^{<\aleph_0}$ is some $J^{<\omega}$ -positive set not containing the empty set.
3. $P(J) \times \text{Cohen}$ separates K

Proof. (2) to (1) is proved in the contrapositive. Suppose (2) fails and \dot{x} is a $P(J)$ -name for an infinite set of natural numbers. We must show that it does not separate the ideal K . Fix a partition $\omega = \bigcup_n z_n$ into infinite sets. For every set $t \in [\omega]^{<\aleph_0}$ and every number n let $D(t, n)$ be the collection of those sets $s \in [\omega]^{<\aleph_0}$ such that $s \neq 0, t \cap s = 0, |s| \in z_n$ and there is a set $a \in J$ such that the condition $\langle t \cup s, a \rangle$ decides the value of the n -th element of the set \dot{x} . Note that for each t, n, s there is just one possible value because such conditions are mutually compatible; denote that value by $f(t, n)(s)$.

Claim 4.10. *For every $t \in [\omega]^{<\aleph_0}$ and every $n \in \omega$ the set $D(n, t)$ is $J^{<\omega}$ -positive.*

By the assumptions, for every t, n the function $g(t, n) : \bigcup_{m>n} D(t, m) \rightarrow \omega$ defined by $g(t, n) = \bigcup_{m>n} f(t, m)$ is not a Rudin-Keisler reduction of the ideal K . There are two cases.

Either there is a set t such that for some number n there is a K -positive set $y \subset \omega$ such that $g(t, n)^{-1}y \in J^{<\omega}$. In such a case, find a set $a \in J$ which intersects every element of the preimage $g(t, n)^{-1}y$. It is not difficult to see that the condition $\langle t, a \rangle$ forces all except the first n elements of the set \dot{x} to miss the set y , and so \dot{x} does not separate the ideal K .

Or else for every set t and for every number n there is a K -small set $y(t, n) \subset \omega$ such that $g(t, n)^{-1}y(t, n) \notin J^{<\omega}$. The sets $t \in [\omega]^{<\aleph_0}$ are now divided into two subcases. In the subcase A, if there are infinitely many numbers n such that the set $y(t, n)$ can be chosen to be a singleton then by the density of the ideal

K there is an infinite set $y(t) \subset \omega$ in the ideal K such that for every number n there is an element $m \in y(t)$ such that $k > n$ and $g(t, 0)^{-1}\{m\} \notin J^{<\omega}$. In the subcase B, if there are only finitely many numbers n such that the set $y(t, n)$ can be chosen to be a singleton then use the fact that the ideal K is a P-ideal and find a set $y(t) \in K$ which modulo finite includes all the sets $y(t, n)$. Finally, find a set $y \in K$ which modulo finite includes all the sets $y(t)$. The proof will be complete once we show that $P(J)$ forces the intersection $\dot{x} \cap \check{y}$ to be infinite.

Well, choose a condition $\langle t, b \rangle$ and a number n . We must find an extension of the condition which forces some number $m > n$ in the set y into the set \dot{x} . First suppose that the set t falls into the subcase A. In such a case, find a number $m > n$ in $y \cap y(t)$ such that $g(t, 0)^{-1}\{m\} \notin J^{<\omega}$. For such a number m there is a set $s \in [\omega]^{<\aleph_0}$ disjoint from b and a condition $\langle t \cup s, a \cup b \rangle \leq \langle t, b \rangle$ which forces $\dot{m} \in \dot{x}$ and we are done. Now suppose that the set t falls into the subcase B. Let $k > n$ be such that the set $y(t, k)$ had to be chosen infinite and $y(t, k) \setminus y$. The $g(t, k)$ -preimage of $y(t, k) \setminus (y \cup n)$ is then $J^{<\omega}$ -positive and there must be a set $s \in [\omega]^{<\aleph_0}$ disjoint from the set b and a condition $\langle t \cup s, b \cup a \rangle$ which forces some number $m \in y(t, k) \setminus n \subset y \setminus n$ into the set \dot{x} .

(2) implies (3) by the previous proposition. The forcing $P(J) \times \text{Cohen}$ separates $J^{<\omega}$, therefore it separates every ideal $J^{<\omega} \restriction b$ where b is a positive set, and also every ideal Rudin-Keisler reducible to it. \square

Perhaps some remarks are in order. The statements (1) and (3) in the previous proposition seem to be very close but perhaps not equivalent. We could not find and verify an example of an ideal J such that the forcing $P(J)$ does not separate the ideal $J^{<\omega}$. Such an ideal J would have to be very strongly inhomogeneous, and in fact any tall summable ideal is a plausible candidate. On the other hand, there are many ideals J such that the forcing $P(J)$ does separate $J^{<\omega}$. A good example of such a behavior is any ideal of the form $J = K^{<\omega}$ for some ideal K or the ideal of nowhere dense subsets of the rationals.

5 Open questions

When the second author's stay in Mexico drew to a close, we decided to finish the work on this paper and leave several interesting questions open.

Question 5.1. Let I be the meager ideal on 2^ω , let $J = tr(I)$. What is the remainder forcing Q_J/P_I ? Similarly for the Lebesgue measure zero ideal.

Question 5.2. Is there a simple preservation criterion on the forcing P_I which is equivalent to the remainder forcing being equal to $\mathcal{P}(\omega)/\text{fin}$?

Question 5.3. Is every proper forcing of the form P_I regularly embeddable into a proper forcing of the form Q_J ? Does every proper forcing of the form P_I have a presentation with the continuous reading of names?

Question 5.4. Assume CH. Is it true that for every ideal J on a countable set, the factor forcing Q_J collapses \aleph_1 if and only if it is \aleph_0 -generated?

Question 5.5. The various definable improper forcings produced in the paper should collapse \mathfrak{c} to \aleph_0 in ZFC. Is this really true?

Question 5.6. For which F_σ P-ideals J, K it is the case that K is not Rudin-Keisler reducible to the ideal $J^{<\omega} \restriction b$ for any $J^{<\omega}$ -positive set $b \subset [\omega]^{<\aleph_0}$?

References

- [1] B. Balcar, F. Hernández Hernández, and M. Hrušák. Combinatorics of dense subsets of the rationals. *Preprint*, 2004.
- [2] T. Bartoszyński and H. Judah. *Set theory. On the structure of the real line*. A K Peters, Ltd., 1995.
- [3] J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki. Decomposing Baire functions. *J. Symb. Logic*, 56:1273–1283, 1991.
- [4] I. Farah. Analytic Hausdorff gaps II: Density zero ideal. *Preprint*, 2004.
- [5] I. Farah and J. Zapletal. Four and more. *Preprint*, 2004.
- [6] T. Jech. *Set theory*. Springer-Verlag, New York, 2003.
- [7] A. S. Kechris. *Classical descriptive set theory*. Springer-Verlag, New York, 1995.
- [8] A. S. Kechris, A. Louveau, and W. H. Woodin. The structure of σ -ideals of compact sets. *Trans. Amer. Math. Soc.*, 301:263–288, 1987.
- [9] K. Mazur. F_σ -ideals and $\omega_1\omega_1^*$ gaps in the Boolean algebra $P(\omega)/I$. *Fund. Math.*, 138:103–111, 1991.
- [10] A. Miller. On relatively Borel and analytic subsets. *Preprint*, 2003.
- [11] S. Solecki. Analytic ideals. *Bull. Symb. Logic*, 2:339–348, 1996.
- [12] S. Solecki. Decomposing borel sets and functions and the structure of Baire class 1 functions. *J. Amer. Math. Soc.*, 11:521–550, 1998.
- [13] S. Solecki. Analytic ideals and their applications. *Ann. Pure Appl. Logic*, 99:51–72, 1999.
- [14] J. Steprāns. Unpublished note.
- [15] J. Steprāns. A very discontinuous Borel function. *J. Symb. Logic*, 58:1268–1283, 1993.
- [16] J. Zapletal. *Descriptive set theory and definable forcing*. Amer. Math. Soc., Providence, 2004. Memoirs of Amer. Math. Soc. 793.